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The bilinear negative Kadomtsev–Petviashvili system and its Kac–Moody–Virasoro symmetry group

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Abstract
The symmetry transformation group of the bilinear negative Kadomtsev–Petviashvili system is studied by means of a direct method. The Kac–Moody–Virasoro-type Lie point symmetry algebra is found to be a special infinitesimal form of the symmetry group.

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1. Introduction

It is interesting that the negative flows or negative integrable hierarchies have been rediscovered several times and have attracted much attention of mathematicians and physicists [1–11]. The first negative flow of the KdV equation

$$Ru_t = 0, \tag{1}$$

where R is the recursion operator of the KdV equation, was first given by Ablowitz *et al* [1] and the bilinear form of the system was provided by Hirota and Satsuma [2].

We have known that the negative KdV equation (1) is related to the Camassa–Holm equation by a hodograph transformation [3]. Some types of exact solutions including the multi-solitons for the negative KdV equation (1) have been found in [4].

Clearly, a special case of the first negative KdV equation is equivalent to the well-known (1+1)-dimensional sine-Gordon (sG) (or sinh-Gordon (ShG)) equation up to a Miura transformation. The Liouville equation can be considered as a commonly equivalent first negative one of the modified KdV equation, the Caudry–Dodd–Gibbon–Sawada–Kotera equation and the Kaup–Kupershmidt equation [5, 9].

In (1+1)-dimensions, there are some other different approaches to finding the integrable negative hierarchies such as the inverse recursion operator approach [8, 9], the gauge transformations [12] and the inner parameter derivative approach [6, 13], etc. Various (1+1)-dimensional negative systems have been obtained and their integrable properties have been studied by some authors.

In [6], a special negative KdV hierarchy is written as

$$\begin{aligned} &u_t = \partial_x \sum_{k=0}^{n} \left[\left(\partial_x^2 + u\right)^{n-k} P_n\right]\left[\left(\partial_x^2 + u\right)^k P_n\right], \\ &\left(\partial_x^2 + u\right)^{n+1} P_n = 0, \qquad n = 1, 2, \ldots. \end{aligned} \tag{2}$$

Similar to the KdV equation, a special negative Kadomtsev–Petviashvili (NKP) hierarchy reads

$$\begin{aligned} &u_t = \partial_x \sum_{k=0}^{n} \left[\left(\partial_x^2 + \alpha\partial_y + u\right)^{n-k} P_n\right]\left[\left(\partial_x^2 - \alpha\partial_y + u\right)^k P_n^*\right], \\ &\left(\partial_x^2 + \alpha\partial_y + u\right)^{n+1} P_n = 0, \qquad \left(\partial_x^2 - \alpha\partial_y + u\right)^{n+1} P_n^* = 0, \qquad n = 1, 2, \ldots, \end{aligned} \tag{3}$$

where α is a constant, which is also derived in [6].

We should remark that the special negative KdV flows (2) can also be derived from the inverse recursion operator of the KdV equation though we still failed to obtain the special NKP flows (3) by using the generalized recursion operator of the KP equation [14].

The first one of (3) (for $n = 0$) is

$$u_t - (\psi\psi^*)_x = 0, \qquad \alpha\psi_y + \left(\partial_x^2 + u\right)\psi = 0, \qquad -\alpha\psi_y^* + \left(\partial_x^2 + u\right)\psi^* = 0. \tag{4}$$

It is interesting and obvious to interpret that the NKP system (4) is just a special case of the squared-eigenfunction symmetry flows of the KP hierarchy described in [15].

In fact, the (2+1)-dimensional (N+M)-component AKNS system [13] is an extension of the squared-eigenfunction symmetry flows of the KP hierarchy. The Painlevé property, the symmetry algebra and the variable separation solutions [16] of the (N+M)-component AKNS system have been studied in [17].

Negative flows of the KP system may have other kinds of extensions such as the Grassmannian description [18] and two-component extensions [19].

By the Miura transformation,

$$u = -\phi_{xx} - \alpha\phi_y - \phi_x^2, \tag{5}$$

$$\psi = \sqrt{\alpha S}\exp\left(-\frac{1}{2}\int \frac{1}{S} s_x\, \mathrm{e}^{2\phi}\, \mathrm{d}x\right), \qquad S \equiv \frac{1}{\alpha}\int (s_x\, \mathrm{e}^{2\phi})_x\, \mathrm{d}y, \tag{6}$$

$$\psi^* = \sqrt{\alpha S}\exp\left(\frac{1}{2}\int \frac{1}{S} s_x\, \mathrm{e}^{2\phi}\, \mathrm{d}x\right), \tag{7}$$

the NKP equation (4) can be changed to a (2+1)-dimensional sinh-Gordon equation [6]

$$\left(\alpha\phi_y + \phi_{xx} + \phi_x^2\right)_{yt} = (s_x\, \mathrm{e}^{2\phi})_{xx}, \tag{8}$$

$$(2\phi_x + \partial_x)\left(\phi_{xt} - \tfrac{1}{2}C_1\, \mathrm{e}^{2\phi} + \tfrac{1}{2}C_2\, \mathrm{e}^{-2\phi}\right) + \alpha\phi_{yt} + \alpha(s\, \mathrm{e}^{2\phi})_x = 0. \tag{9}$$

Remark. The Miura transformation (5) is completely the same as the transformation between the usual KP equation and the modified KP equation. From equations (3) and (4), it is evident that if α (or y) is pure imaginary then ψ^* is just the complex conjugate of ψ. ψ^* given by (7) is not explicitly the complex conjugate of ψ of (6) because the Miura transformation (5) itself is not explicitly complex conjugate. The real condition of u and the complex conjugate condition of ψ^* with respect to ψ read

$$\phi^*_{xx} + \alpha^*\phi^*_{y^*} + (\phi^*_x)^2 = \phi_{xx} + \alpha\phi_y + \phi_x^2, \tag{10}$$

$$[\ln(\alpha^* S^* - \alpha S)]_x + \frac{1}{S^*} s^*_x \,\mathrm{e}^{2\phi^*} + \frac{1}{S} s_x \,\mathrm{e}^{2\phi} = 0. \tag{11}$$

Obviously, the system (8) and (9) will reduce to the (1+1)-dimensional sinh-Gordon equation when the field ϕ is y-independent and $s = 0$.[4]

Without loss of generality, we can choose

$$2\phi = \omega, \qquad C_1 = C_2 = \tfrac{1}{2}, \qquad \alpha = 1, \qquad s = \theta,$$

and thus the system of equations (8) and (9) becomes

$$(\omega_{xt} - \sinh\omega)_x + \omega_x(\omega_{xt} - \sinh\omega) + \omega_{yt} = -2(\theta\,\mathrm{e}^{\omega})_x, \tag{12}$$

$$2(\theta_x\,\mathrm{e}^{\omega})_{xx} + \left(\omega_y + \omega_{xx} + \tfrac{1}{2}\omega_x^2\right)_{yt} = 0, \tag{13}$$

which is different from the complicated known one [21] because of the space $\{x, y\}$ asymmetric property. For the ShG system (12) and (13), the authors of [20] have presented us its bilinear form

$$\left(D_y + D_x^2\right) f \cdot g = 0, \tag{14}$$

$$D_t\left(D_y + D_x^2\right) f \cdot g = 0, \tag{15}$$

in which $f \equiv f(x, y, t)$, $g \equiv g(x, y, t)$, and the operators D_t, D_x and D_y are defined as $D_x^m D_y^n D_t^k f \cdot g = \partial_a^m \partial_b^n \partial_c^k f(x + a, y + b, t + c) g(x - a, y - b, t - c)\big|_{a=b=c=0}$, which were first introduced by Hirota [27] with f, g, ω and θ being related by

$$\omega(x, y, t) = 2\ln\frac{f(x, y, t)}{g(x, y, t)} \tag{16}$$

$$\theta(x, y, t) = 2\int_{-\infty}^{x} [\ln g(\xi, y, t)]_{yt} \left[\frac{g(\xi, y, t)}{f(\xi, y, t)}\right]^2 \mathrm{d}\xi. \tag{17}$$

We should interpret that the bilinear system may have some different names because it may have some different types of nonlinear equations, such as the negative KP equation, the (2+1)-dimensional (1+1)-component AKNS system, the (2+1)-dimensional Broer–Kaup system [22] and the (2+1)-dimensional sinh-Gordon equation.

It should also be mentioned that the NKP system possesses a more general form

$$\left(D_y + D_x^2\right) f \cdot g = 0, \tag{18}$$

$$D_t\left(D_y + D_x^2\right) f \cdot g + q(y, t) D_x f \cdot g = 0, \tag{19}$$

if a nonzero boundary term, $\theta(-\infty, y, t) = q(y, t)$ is added to the transformation (17). Equations (18) and (19) are just equations (10) and (11) of [30].

[4] Note in this special y-independent case, the original eigenfunction variables ψ and $\psi^* = \psi$ should be related to ϕ by $\psi_{xx} = (\phi_{xx} + \phi_x^2)\psi$ instead of the undefined equations (6) and (7).

Actually, the bilinear form given by Hu *et al* [20] is corresponding to $q(y,t) = 2$, i.e., equations (5) and (6) of [30]. On the other hand, if one takes $y = x$, $q = 0$ and/or $q = 1$ in (18) and (19), it is just two known bilinear forms of the classical Bossinesq equation[24]. In this paper, we focus our attention on the $q = 0$ case for simplicity.

The knowledge of the symmetries is very useful to enhance our understanding of complex physical phenomena, to simplify and even completely solve the complicated problems. Furthermore, the study of symmetries has been manifested as one of the most important and powerful methods in almost every branch of science especially in physics and mathematics. It is particularly fundamental to find the symmetries of a nonlinear equation and then to construct its Lie algebra in the development of the theory of the integrable systems because of the existence of infinitely many symmetries.

The symmetry groups of a nonlinear system are commonly obtained by using the Lie's first fundamental theorem; however, it is rather difficult to be fulfilled due to complicated calculations. Recently, Lou and Ma proposed a direct method for deriving symmetry groups of a nonlinear system in [23]. The new direct method can be used not only to find the equivalent while much simpler and explicit Lie point symmetry groups, but to present the non-Lie symmetry groups as well.

In section 2 of this paper, we apply the direct method developed in [23] to obtain the transformation group of the NKP equations (14) and (15) (or equivalently, (20) and (21)), and the Lie point symmetries are presented in section 3.

2. Transformation group by the direct method

The system of the differential equations (14) and (15) is equivalent to

$$f_y g - f g_y + f_{xx} g - 2 f_x g_x + f g_{xx} = 0, \tag{20}$$

$$(g\partial_t - g_t)(f_y + f_{xx}) + (f\partial_t - f_t)(g_y - g_{xx}) - 2(f_{xt} g_x - f_x g_{xt}) = 0. \tag{21}$$

To find a complete point symmetry transformation group of (20) and (21), i.e. (14) and (15), one should find the general transformations in the following form,

$$\begin{aligned} f &= f_1(x, y, t, F(\xi, \eta, \tau), G(\xi, \eta, \tau)), \\ g &= g_1(x, y, t, F(\xi, \eta, \tau), G(\xi, \eta, \tau)), \end{aligned} \tag{22}$$

where ξ, η and τ are functions of x, y, t, f and g, $F \equiv F(\xi, \eta, \tau)$ and $G \equiv G(\xi, \eta, \tau)$ are also solutions of the bilinear NKP equation in the variables ξ, η and τ, i.e.,

$$F_\eta G - F G_\eta + F_{\xi\xi} G - 2 F_\xi G_\xi + F G_{\xi\xi} = 0, \tag{23}$$

$$(G\partial_\tau - G_\tau)(F_\eta + F_{\xi\xi}) + (F\partial_\tau - F_\tau)(G_\eta - G_{\xi\xi}) - 2(F_{\xi\tau} G_\xi - F_\xi G_{\xi\tau}) = 0. \tag{24}$$

Fortunately, similar to the usual KP case in [23], we can prove that for the bilinear NKP system it is enough to take

$$f = \beta_1 F(\xi, \eta, \tau), \qquad g = \beta_2 G(\xi, \eta, \tau) \tag{25}$$

instead of (22), where β_1, β_2 and ξ, η, τ are functions of $\{x, y, t\}$.

To prove the conclusion (25), one should submit the general expression (22) to the bilinear equations (20) and (21). After eliminating G_η and $F_{\xi\xi\tau}$ and their higher derivatives via (23) and (24) and vanishing all the coefficients of the different terms of the derivatives of the functions F and G, one can get 409 complicated determining equations for five functions $f_1 \equiv f_1(x, y, t, F, G)$, $g_1 \equiv g_1(x, y, t, F, G)$, ξ, η and τ. Five of them read as

$$\xi_x^2\tau_t^2(g_1 f_{1FF} - f_1 g_{1FF}) = 0, \qquad \xi_x^2\tau_t^2(g_1 f_{1GG} - f_1 g_{1GG}) = 0,$$
$$\xi_x^2(g_1 f_{1FF} + f_1 g_{1FF} - 2 f_{1F} g_{1F}) = 0, \qquad \xi_x^2(g_1 f_{1GG} + f_1 g_{1GG} - 2 f_{1G} g_{1G}) = 0,$$
$$\eta_y[F(g_1 f_{1F} - f_1 g_{1F}) + G(f_1 g_{1G} - g_1 f_{1G})] = 0$$

which have only three types of solutions:

Case 1.

$$f_1 = \beta_1(x, y, t)F, \qquad g_1 = \beta_2(x, y, t)G.$$

Case 2.

$$f_1 = a_1(x, y, t)G, \qquad g_1 = a_2(x, y, t)F.$$

Case 3.

$$f_1 = b(x, y, t)g_1$$

for $\xi_x\eta_y\tau_t \neq 0$. When $\xi_x\eta_y\tau_t = 0$, we can reasonably prove that one cannot find any nontrivial symmetry transformations.

If substituting the third case, $f_1 = b(x, y, t)g_1$, into the remaining determining equations, one can find that $f_{1F} = f_{1G} = 0$ which will also result in no symmetry transformations.

The first case is just the result we want to prove while the second result is only related to the discrete symmetry transformation of the original model $\{F, G, y\} \to \{G, F, -y\}$. This situation will be included in the final group transformation theorem 1 (see later). Therefore, the conclusion (25) is proved.

Substituting (25) into equation (20) with F and G satisfying the negative KP equation yields:

$$2\beta_1\beta_2\xi_x\tau_x FGF_{\xi\tau} + W_1(x, y, t, F, F_\xi, \ldots, F_\eta, \ldots, G_\xi, G_\eta, \ldots, G_{\xi\eta}) = 0, \tag{26}$$

where W_1 is a complicated $F_{\xi\tau}$ (and its higher order derivatives) independent function. Equation (26) exists for an arbitrary solution F only when all the coefficients of the polynomials of the derivatives of F and G are zero.

Obviously, β_1 and β_2 should not be zero, and ξ is the transformation of x which requires $\xi_x \neq 0$. Since there is no nontrivial solution for $\tau_x \neq 0$, we must have

$$\tau_x = 0, \qquad \text{i.e.} \quad \tau \equiv \tau(y, t). \tag{27}$$

Under the condition (27), equation (26) is reduced to

$$2\eta_x\xi_x\beta_2\beta_1 F^2 G_{\xi\eta} + W_2(x, y, t, F, F_\xi, \ldots, F_\eta, \ldots, G_\xi, G_\eta, \ldots, G_{\xi\tau}) = 0, \tag{28}$$

with W_2 a complicated $G_{\xi\eta}$ independent function. Obviously, the only possible case to vanish the coefficient of $G_{\xi\eta}$ is

$$\eta_x = 0, \qquad \text{i.e.} \quad \eta \equiv \eta(y, t). \tag{29}$$

Under the above condition, a simplified form of equation (28) can now be completely written:

$$\begin{aligned}\big\{[2\xi_x(\beta_1\beta_{2x} - \beta_{1x}\beta_2) + \beta_1\beta_2(\xi_{xx} - \xi_y)]G_\xi + \beta_1\beta_2\big(\xi_x^2 - \eta_y\big)G_\eta - \beta_1\beta_2\tau_y G_\tau \\ + (\beta_{1xx}\beta_2 - 2\beta_{1x}\beta_{2x} + \beta_1\beta_{2xx} + \beta_{1y}\beta_2 - \beta_1\beta_{2y})G\big\}F^2 + \big\{[\beta_1\beta_2(\xi_{xx} + \xi_y) \\ + \xi_x(\beta_{1x}\beta_2 - \beta_1\beta_{2x})]F_\xi + \beta_1\beta_2\big(\eta_y - \xi_x^2\big)F_\eta + \beta_1\beta_2\tau_y F_\tau\big\}FG = 0.\end{aligned} \tag{30}$$

Equation (30) is true for arbitrary solutions F and G only when all the coefficients of the polynomials of the derivatives of F and G are zero, which leads to a system of determining equations of the unknown functions

$$\beta_1\beta_2\tau_y = 0, \tag{31}$$

$$\beta_1\beta_2\left(\xi_x^2 - \eta_y\right) = 0, \tag{32}$$

$$\beta_1\beta_2(\xi_{xx} + \xi_y) + \xi_x(\beta_{1x}\beta_2 - \beta_1\beta_{2x}) = 0, \tag{33}$$

$$2\xi_x(\beta_1\beta_{2x} - \beta_{1x}\beta_2) + \beta_1\beta_2(\xi_{xx} - \xi_y) = 0. \tag{34}$$

It is not difficult to obtain the general solutions of the determining equations (31)–(34). The results are

$$\tau \equiv \tau(t), \qquad \eta \equiv \eta(y,t), \tag{35}$$

$$\xi = \xi_1(y,t)x + f_1(y,t), \tag{36}$$

$$\eta_y = \xi_1(y,t)^2, \tag{37}$$

$$\beta_1 = b_1(y,t)\beta_2 \exp\left[-\frac{x}{4\xi_1}(2f_{1y} + \xi_{1y}x)\right] \tag{38}$$

$$\begin{aligned}\beta_2 = b_3(y,t)\exp\Bigg[&-\frac{1}{96}\frac{\xi_1\xi_{1yy} - 2\xi_{1y}^2}{\xi_1^2}x^4 + \frac{1}{24}\frac{\xi_1\xi_{1yy} - 2f_{1y}\xi_{1y}}{\xi_1^2}x^3 \\ &- \frac{1}{16}\frac{b_1 f_{1y}^2 + 4b_{1y}\xi_1^2 - 2b_1\xi_{1y}\xi_1}{\xi_1^2}x^2 - b_2(y,t)x\Bigg],\end{aligned} \tag{39}$$

where ξ_1, f_1, b_1, b_2 and b_3 are functions of $\{y,t\}$ which should be further determined through equation (21). Substituting (25) and the known results into equation (21) we have

$$2\beta_1^2\beta_2^2\eta_t\xi_1^2 GFF_{\xi\xi\eta} + W_3(x,y,t,F,F_\xi,\ldots,F_\eta,\ldots,G_\xi,G_\eta,\ldots,G_{\xi\tau}) = 0, \tag{40}$$

where W_3 is an $F_{\xi\xi\eta}$ independent function. Evidently, (40) is true only for the coefficient of $F_{\xi\xi\eta}$ being zero which means only

$$\eta_t = 0, \qquad \text{i.e.} \quad \eta \equiv \eta(y), \tag{41}$$

for ξ_1 cannot be zero on account of (36). Thus equation (40) reduces further to

$$2\beta_1^2\beta_2^2\xi_1^2(\xi_{1t}x + f_{1t})GFF_{\xi\xi\xi} + W_4(x,y,t,F,F_\xi,\ldots,F_\eta,\ldots,G_\xi,G_\eta,\ldots,G_{\xi\tau}) = 0, \tag{42}$$

with W_4 being an $F_{\xi\xi\xi}$ independent function. Thus there is no choice but

$$\begin{aligned}&\xi_{1t} = 0, \qquad \text{i.e.} \quad \xi_1(y,t) \equiv \xi_1(y),\\ &f_{1t} = 0, \qquad \text{i.e.} \quad f_1(y,t) \equiv f_1(y),\end{aligned} \tag{43}$$

which lead equation (42) to be

$$\begin{aligned}&b_3\left(\xi_1 b_1 b_{1t} b_{1yy} - 2\xi_1 b_{1t} b_{1y}^2 + 2\xi_1 b_1 b_{1y} b_{1yt} - \xi_1 b_1^2 b_{1yyt} + 2\xi_{1y} b_1^2 b_{1yt}\right) GFx^2 \\ &\quad - b_1 b_3^2 d\left(-4b_1^2\xi_{1y}b_{2t} + 4b_1^2 b_{2yt}\xi_1 - 2b_{1yt}f_{1y}b_1 + 2b_{1t}b_{1y}f_{1y}\right)GFx \\ &\quad - b_1\left(-4b_1^2 b_3\xi b_{3yt} - 4b_1^2 b_3^2 f_{1y} b_{2t} + 4b_1^2 b_{3y} b_{3t}\xi \right.\\ &\quad \left. - 2b_3^2 b_{1yt} b_1\xi_1 + 2b_3^2\xi_1 b_{1t} b_{1y}\right)GF \\ &\quad - 4\xi_1^2 b_1 b_3^2(b_{1y}b_{1t} - b_1 b_{1yt})(G_\xi F - F_\xi G)x \\ &\quad + 2\xi_1^2 b_1^3 b_3^2 4b_{2t}(G_\xi F - F_\xi G) = 0,\end{aligned} \tag{44}$$

and then we have the further determining equations

$$4b_{2t} = 0, \tag{45}$$

$$b_{1y}b_{1t} - b_1 b_{1yt} = 0, \tag{46}$$

$$-2\xi_1 b_{1t} b_{1y}^2 + \xi_1 b_1 b_{1t} b_{1yy} + 2\xi_1 b_1 b_{1y} b_{1yt} - \xi_1 b_1^2 b_{1yyt} + 2\xi_{1y} b_1^2 b_{1yt} = 0, \tag{47}$$

$$-4b_1^2 \xi_{1y} b_{2t} + 4b_1^2 b_{2yt}\xi_1 - 2b_{1yt} f_{1y} b_1 + 2b_{1t} b_{1y} f_{1y} = 0, \tag{48}$$

$$-4b_1^2 b_3 \xi b_{3yt} - 4b_1^2 b_3^2 f_{1y} b_{2t} + 4b_1^2 b_{3y} b_{3t}\xi - 2b_3^2 b_{1yt} b_1 \xi_1 + 2b_3^2 \xi_1 b_{1t} b_{1y} = 0. \tag{49}$$

The general solutions of equations (45)–(49) are

$$b_1(y,t) = h_2(y) r_0(t), \tag{50}$$

$$b_2(y,t) \equiv b_2(y), \tag{51}$$

$$b_3(y,t) = h_1(y)\tau_0(t), \tag{52}$$

with h_1, h_2 and b_2 being arbitrary functions of y, and r_0 and τ_0 being arbitrary functions of t.

In summary, after considering the possible discrete transformation $\{y, f, g\} \to \{-y, g, f\}$ discussed before, the following theorem holds:

Theorem 1. *If $\{F = F_0(x,y,t), G = G_0(x,y,t)\}$ is a solution of the bilinear negative KP equations (20) and (21), then so are $\{F_1, G_1\}$ and $\{F_2, G_2\}$ with*

$$\begin{aligned} F_1 = r_0(t)\tau_0(t)h_1(y)h_2(y)\exp\bigg[&\frac{x^4}{96\xi_1^2}\left(\xi_{1yy}\xi_1 - 2\xi_{1y}^2\right) + \frac{x^3}{24\xi_1^2}(f_{1yy}\xi_1 - 2f_{1y}\xi_{1y}) \\ &- \frac{x^2}{16h_2\xi_1^2}\left(f_{1y}^2 h_2 + 2\xi_{1y}\xi_1 h_2 + 4h_{2y}\xi_1^2\right) - \frac{x}{2\xi_1}(2\xi_1 b_2 + f_{1y})\bigg] F_0(\xi_1(y)x \\ &+ f_1(y), \eta(y), \tau(t)), \end{aligned} \tag{53}$$

$$\begin{aligned} G_1 = \tau_0(t)h_1(y)\exp\bigg[&\frac{x^4}{96\xi_1^2}\left(\xi_{1yy}\xi_1 - 2\xi_{1y}^2\right) + \frac{x^3}{24\xi_1^2}(f_{1yy}\xi_1 - 2f_{1y}\xi_{1y}) \\ &- \frac{x^2}{16h_2\xi_1^2}\left(f_{1y}^2 h_2 - 2\xi_{1y}\xi_1 h_2 + 4h_{2y}\xi_1^2\right) - b_2(y)x\bigg] \\ &\times G_0(\xi_1(y)x + f_1(y), \eta(y), \tau(t)), \end{aligned} \tag{54}$$

$$\begin{aligned} F_2 = r_0(t)\tau_0(t)h_1(y)h_2(y)\exp\bigg[&\frac{x^4}{96\xi_1^2}\left(\xi_{1yy}\xi_1 - 2\xi_{1y}^2\right) + \frac{x^3}{24\xi_1^2}(f_{1yy}\xi_1 - 2f_{1y}\xi_{1y}) \\ &- \frac{x^2}{16h_2\xi_1^2}\left(f_{1y}^2 h_2 - 2\xi_{1y}\xi_1 h_2 - 4h_{2y}\xi_1^2\right) - \frac{x}{2\xi_1}(2\xi_1 b_2 - f_{1y})\bigg] G_0(\xi_1(y)x \\ &+ f_1(y), \eta(y), \tau(t)), \end{aligned} \tag{55}$$

$$\begin{aligned} G_2 = \tau_0(t)h_1(y)\exp\bigg[&\frac{x^4}{96\xi_1^2}\left(\xi_{1yy}\xi_1 - 2\xi_{1y}^2\right) + \frac{x^3}{24\xi_1^2}(f_{1yy}\xi_1 - 2f_{1y}\xi_{1y}) \\ &- \frac{x^2}{16h_2\xi_1^2}\left(f_{1y}^2 h_2 + 2\xi_{1y}\xi_1 h_2 - 4h_{2y}\xi_1^2\right) - b_2(y)x\bigg] \\ &\times F_0(\xi_1(y)x + f_1(y), \eta(y), \tau(t)), \end{aligned} \tag{56}$$

where ξ_1, b_2, h_1, h_2 and f_1 are arbitrary functions of y, r_0, τ_0 and $\tau \equiv \tau(t)$ are arbitrary functions of t, while $\eta \equiv \eta(y)$ and ξ_1 are linked by

$$\eta_y = \xi_1^2, \tag{57}$$

or equivalently

$$\xi_1 = \pm\sqrt{\eta_y}. \tag{58}$$

The group theorem 1 indicates the full Lie point symmetry group, $\mathcal{G}$, of the model can be divided into two parts

$$\mathcal{G} = \mathcal{S} \otimes \mathcal{D},$$

where $\mathcal{S}$ stands for the Lie symmetry group expressed by (53) and (54) with the negative sign of (58) while $\mathcal{D}$ stands for a discrete group,

$$\mathcal{D} = \{I, \sigma_x, \sigma_y, \sigma_x\sigma_y\},$$

where I is an identity transformation and σ_x and σ_y are related to two discrete transformations

$$\sigma_x : \{x, y, t, f, g\} \longrightarrow \{-x, y, t, f, g\},$$

and

$$\sigma_y : \{x, y, t, f, g\} \longrightarrow \{x, -y, t, g, f\}.$$

In other words the full Lie point symmetry group of the model is divided into four sectors $\mathcal{S}, \sigma_x\mathcal{S}, \sigma_y\mathcal{S}$ and $\sigma_x\sigma_y\mathcal{S}$. This kind of phenomena has also been found for some other models [23, 25]. One should be careful in the practical application of theorem 1, the first step is to change the independent variables $\{x, y, t\}$ of the original solution $\{F_0, G_0\}$ to $\{\xi, \eta, \tau\} \equiv \{\xi_1(y)x + f_1(y), \eta(y), \tau(t)\}$ and then multiply the changed functions by some necessary factors given in the theorem.

Applying the theorem to some simple exact solutions without arbitrary functions, one may obtain some types of novel generalized solutions with some arbitrary functions. In the following, we just present one special solution example.

Example 1. It is quite trivial that the bilinear equation system of equations (14) and (15) possesses a special simple solution

$$F_0 = 1, \tag{59}$$

$$G_0 = 1 + \tau_1\,\mathrm{e}^{k_1x+k_1^2y} + \tau_2\,\mathrm{e}^{k_2x+k_2^2y}, \tag{60}$$

where k_1, k_2 are arbitrary constants and τ_1, τ_2 are arbitrary functions of t. Correspondingly, the special solutions of the ShG equations (12) and (13) have the form

$$\omega_0 = -2\ln(1 + \tau_1\,\mathrm{e}^{k_1x+k_1^2y} + \tau_2\,\mathrm{e}^{k_2x+k_2^2y}), \tag{61}$$

$$\begin{aligned}\theta_0 = {} & 2k_1\tau_{1t}\,\mathrm{e}^{k_1x+k_1^2y} + 2k_2\tau_{2t}\,\mathrm{e}^{k_2x+k_2^2y} \\ & - 2(k_1 - k_2)(\tau_1\tau_{2t} - \tau_2\tau_{1t})\,\mathrm{e}^{(k_1+k_2)x+(k_1^2+k_2^2)y}.\end{aligned} \tag{62}$$

Using the transformation theorem to the above special solution we have the following new special solution of the NKP equation,

$$\begin{aligned}F = {} & r_0(t)\tau_0(t)h_1(y)h_2(y)\exp\left[\frac{x^4}{96\xi_1^2}\left(\xi_{1yy}\xi_1 - 2\xi_{1y}^2\right) + \frac{x^3}{24\xi_1^2}(f_{1yy}\xi_1 - 2f_{1y}\xi_{1y})\right. \\ & \left. - \frac{x^2}{16h_2\xi_1^2}\left(f_{1y}^2h_2 + 2\xi_{1y}\xi_1h_2 + 4h_{2y}\xi_1^2\right) - \frac{x}{2\xi_1}(2\xi_1b_2 + f_{1y})\right],\end{aligned} \tag{63}$$

$$
\begin{aligned}
G = \tau_0(t)h_1(y)\exp\Bigg[&\frac{x^4}{96\xi_1^2}\left(\xi_{1yy}\xi_1 - 2\xi_{1y}^2\right) + \frac{x^3}{24\xi_1^2}(f_{1yy}\xi_1 - 2f_{1y}\xi_{1y}) \\
&- \frac{x^2}{16h_2\xi_1^2}\left(f_{1y}^2h_2 - 2\xi_{1y}\xi_1h_2 + 4h_{2y}\xi_1^2\right) - b_2(y)x\Bigg] \\
&\times (1 + \tau_1\,\mathrm{e}^{k_1(\xi_1x+f_1)+k_1^2\eta} + \tau_2\,\mathrm{e}^{k_2(\xi_1x+f_1)+k_2^2\eta}),
\end{aligned}
\tag{64}
$$

where we have rewritten $\tau_1(\tau(t))$ and $\tau_2(\tau(t))$ as $\tau_1(t)$ and $\tau_2(t)$ because the original τ_1 and τ_2 are arbitrary functions of t.

Accordingly, the new solution for the ShG system reads

$$
\omega = 2\ln\frac{r_0h_2\exp\left[-\frac{1}{4}x\xi_1^{-1}(\xi_{1y}x + 2f_{1y})\right]}{1 + \tau_1\,\mathrm{e}^{k_1(\xi_1x+f_1)+k_1^2\eta} + \tau_2\,\mathrm{e}^{k_2(\xi_1x+f_1)+k_2^2\eta}},
\tag{65}
$$

$$
\begin{aligned}
\theta = \frac{2\xi_1\exp\left[\frac{1}{2}x\xi_1^{-1}(\xi_{1y}x + f_{1y})\right]}{h_2^2r_0^2}&\left\{k_1\tau_{1t}\mathrm{e}^{k_1(\xi_1x+f_1)+k_1^2\eta} + k_2\tau_{2t}\,\mathrm{e}^{k_2(\xi_1x+f_1)+k_2^2\eta}\right. \\
&\left. - (k_1 - k_2)(\tau_1\tau_{2t} - \tau_2\tau_{1t})\,\mathrm{e}^{(k_1+k_2)(\xi_1x+f_1)+(k_1^2+k_2^2)\eta}\right\}.
\end{aligned}
\tag{66}
$$

We believe that this type of solution can also be obtained by means of other methods (say, the dressing method), but the detailed procedure might be rather complicated.

3. The Kac–Moody–Virasoro structure of the Lie point symmetry algebra

In the traditional Lie group theory, one always tries to find the Lie point symmetries first and then use the Lie's first fundamental theorem to obtain the symmetry transformation group. Conversely, in the last section we are fortunate to obtain the symmetry transformation group in the first place by a simple direct method. Once the transformation group is known, the Lie point symmetries and the related Lie symmetry algebra can be obtained straightforwardly by a more simple limiting procedure (which can also be obtained by other standard methods in [30]).

For the bilinear NKP system (14) and (15), the corresponding Lie point symmetries can be derived from the symmetry group transformation theorem presented in the last section by setting

$$
\begin{aligned}
&\eta(y) = y + \epsilon e(y), && \tau(t) = t + \epsilon C(t), \\
&f_1(y) = \epsilon d(y), && r_0(t) = 1 - \epsilon A(t), \\
&b_2(y) = \epsilon(b(y) - \tfrac{1}{4}d'(y)), && h_1(y) = 1 - \epsilon(a(y) - \tfrac{1}{2}c(y)), \\
&\tau_0(t) = 4\epsilon B(t), && h_2(y) = 1 - \epsilon c(y),
\end{aligned}
$$

and

$$
\xi_1(y) = \sqrt{1 + \epsilon e'(y)},
$$

with ϵ being an infinitesimal parameter, primes denoting the derivatives with respect to y and dots over the functions (will appear later) denoting the derivatives with respect to t. Under the above selections, (53) and (54) become

$$
f = F + \epsilon\sigma^f(F) + O(\epsilon^2),
$$

$$
\sigma^f = C(t)F_t + e(y)F_y + \left(d(y) + \frac{1}{2}xe'(y)\right)F_x + \left[\frac{x^4}{192}e'''(y) + \frac{x^3}{24}d''(y)\right.
$$

$$+\frac{x^2}{4}\left(c'(y)-\frac{1}{4}e''(y)\right)-\left(\frac{1}{4}d'(y)+b(y)\right)x$$
$$-a(y)-A(t)-B(t)-\frac{1}{2}c(y)\bigg]F, \tag{67}$$

$$g=G+\epsilon\sigma^g(G)+O(\epsilon^2),$$
$$\sigma^g=C(t)G_t+e(y)G_y+\left(d(y)+\frac{1}{2}xe'(y)\right)G_x+\left[\frac{x^4}{192}e'''(y)+\frac{x^3}{24}d''(y)\right.$$
$$\left.+\frac{x^2}{4}\left(c'(y)+\frac{1}{4}e''(y)\right)+\left(\frac{1}{4}d'(y)-b(y)\right)x-a(y)-B(t)+\frac{1}{2}c(y)\right]G, \tag{68}$$

which means that the bilinear NKP system possesses the Lie point symmetries

$$\sigma=\begin{pmatrix}\sigma^f\\ \sigma^g\end{pmatrix}. \tag{69}$$

The equivalent vector expression of the above symmetry reads

$$V=V_1(A(t))+V_2(B(t))+V_3(C(t))$$
$$+W_1(a(y))+W_2(b(y))+W_3(c(y))+W_4(d(y))+W_5(e(y)), \tag{70}$$

with

$$V_1(A(t))=A(t)F\frac{\partial}{\partial F}, \tag{71}$$

$$V_2(B(t))=B(t)\left(F\frac{\partial}{\partial F}+G\frac{\partial}{\partial G}\right), \tag{72}$$

$$V_3(C(t))=C(t)\frac{\partial}{\partial t}, \tag{73}$$

$$W_1(a(y))=a(y)\left(F\frac{\partial}{\partial F}+G\frac{\partial}{\partial G}\right), \tag{74}$$

$$W_2(b(y))=xb(y)\left(F\frac{\partial}{\partial F}+G\frac{\partial}{\partial G}\right), \tag{75}$$

$$W_3(c(y))=c(y)F\frac{\partial}{\partial F}-\frac{1}{4}(2c(y)+x^2c'(y))\left(F\frac{\partial}{\partial F}+G\frac{\partial}{\partial G}\right), \tag{76}$$

$$W_4(d(y))=d(y)\frac{\partial}{\partial x}+\frac{1}{2}xd'(y)F\frac{\partial}{\partial F}-\frac{1}{24}(6xd'(y)+x^3d''(y))\left(F\frac{\partial}{\partial F}+G\frac{\partial}{\partial G}\right), \tag{77}$$

$$W_5(e(y))=e(y)\frac{\partial}{\partial y}+\frac{1}{2}xe'(y)\frac{\partial}{\partial x}-\frac{x^4}{192}e'''(y)\left(F\frac{\partial}{\partial F}+G\frac{\partial}{\partial G}\right)$$
$$+\frac{x^2}{16}e''(y)\left(F\frac{\partial}{\partial F}-G\frac{\partial}{\partial G}\right). \tag{78}$$

It is easy to verify that the symmetries $V_i, W_j, i=1,2,3, j=1,2,\ldots,5$, constitute an infinite-dimensional Kac–Moody–Virasoro [29] type symmetry algebra $\boldsymbol{S}$ with the following nonzero commutation relations:

$$[V_3(C), V_1(A)] = V_1(C\dot{A}), \tag{79}$$

$$[V_3(C), V_2(B)] = V_2(C\dot{B}), \tag{80}$$

$$[V_3(C_1), V_3(C_2)] = V_3(C_1\dot{C}_2 - C_2\dot{C}_1), \tag{81}$$

$$[W_4(d), W_2(b)] = W_1(bd), \tag{82}$$

$$[W_3(c), W_4(d)] = \tfrac{1}{2}W_1(dc'), \tag{83}$$

$$[W_4(d_1), W_4(d_2)] = \tfrac{1}{2}W_3(d_1d_2' - d_2d_1'), \tag{84}$$

$$[W_5(e), W_1(a)] = W_1(ea'), \tag{85}$$

$$[W_5(e), W_2(b)] = W_2\left(eb' + \tfrac{1}{2}be'\right), \tag{86}$$

$$[W_5(e), W_3(c)] = W_3(ec'), \tag{87}$$

$$[W_5(e), W_4(d)] = W_4(ed'), \tag{88}$$

$$[W_5(e), W_4(d)] = W_4\left(ed' - \tfrac{1}{2}de'\right), \tag{89}$$

$$[W_5(e_1), W_5(e_2)] = W_5(e_1e_2' - e_2e_1'). \tag{90}$$

It should be emphasized that the algebra is infinite dimensional because the generators V_1, V_2, V_3, W_1, W_2, W_3 and W_4 all contain *arbitrary* functions. The algebra is closed because all the commutators can be expressed by the generators belonging to the generator set usually with different functions and the generators contained *different* functions belonging to the set.

From (79)–(90), we easily see that $\boldsymbol{S}_1 = \{V_1, V_2\}$, $\boldsymbol{S}_2 = \{W_1, W_2, W_3, W_4\}$, $\boldsymbol{S}_3 = \{V_3\}$ and $\boldsymbol{S}_4 = \{W_5\}$ are four subalgebras of the whole Lie point symmetry algebra $\boldsymbol{S}$, $\boldsymbol{S}_1$ is a commutative algebra, $\boldsymbol{S}_2$ is a special type of Kac–Moody algebra, and $\boldsymbol{S}_3$ and $\boldsymbol{S}_4$ are two generalized Witt algebras which are also called centerless Virasoro algebras.

If one applies the traditional Lie group theory to the bilinear NKP system (14) and (15), the same Lie algebra can be found given in this section [30]. However, if one tries to obtain the Lie point symmetry group starting from the Lie symmetry algebra by using the Lie's first fundamental theorem, we believe a much more complicated form is presented.

4. Summary and discussion

In summary, the bilinear NKP equation is studied, which is an integrable (2+1)-dimensional extension of the well-known sinh-Gordon (or sine-Gordon) equation and the Broer–Kaup equation. The model possesses various interesting and good properties, such as infinitely many symmetries. In this paper, we found that the model possesses two infinite-dimensional Kac–Moody–Virasoro-type symmetry algebras constituted by $\{V_1, V_2, V_3\}$ and $\{W_1, W_2, W_3, W_4, W_5\}$, respectively.

In the traditional Lie symmetry group theory, one usually first finds the Lie point symmetry algebra by the prolongation approach and then obtains the Lie symmetry group via solving the related initial value problem based on the Lie's first fundamental theorem. However, it is rather difficult to find the general Lie point symmetry algebras and the Lie symmetry groups by means of the traditional method. Furthermore, even if the symmetry transformation group can be derived via the traditional method, the results usually are too complicated to be used to yield the general group invariant solutions from a special one.

In this paper, applying the direct method provided in [23] to the bilinear NKP equation, the simple symmetry transformation group is presented first and then the related Lie point symmetries (which can also be obtained by other standard methods [30]) are provided simply by a limiting procedure.

Because of the importance of the integrable models in the nonlinear science, it is worth paying more attention to the NKP equation, the NKP hierarchy and other negative integrable hierarchies.

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